T_K = $\dfrac{3.413\,K\,(nFy_i^{\circ})^2}{2M_{av}c_p}\left(\beta_1+\dfrac{\alpha_2}{\sigma}\right)$, °F.

T_s = $\dfrac{y_i''T_m(-\Delta S)}{M_{av}c_p}+\dfrac{3.413\,nFy_i^{\circ}\beta_o}{M_{av}c_p}$, °F.

T_{ad} = an adiabatic reaction temperature = $\dfrac{\psi a_3}{Gc_p}$ or $\dfrac{\psi_o a_3}{Gc_p}$ for rate varying with temperature, °F.

T_m = average absolute temperature of the electrolyte, °R.

T_A = inlet pack air temperature, °F.

T_w = wall temperature, °F.

t = temperature of the air stream, °F.

t' = temperature of the fuel stream, °F.

t_o = inlet temperature of the air stream, °F.

t_o' = inlet temperature of the fuel stream, °F.

v = linear velocity, ft./hr.

x,y,z = length coordinates, ft.

$\bar{x},\bar{y},\bar{z}$ = $\dfrac{x}{a_1},\dfrac{y}{a_2},\dfrac{z}{a_3}$, respectively

y_i = mole fraction of the controlling species

y_i° = mole fraction of the controlling species entering the fuel cell

y_i' = mole fraction of the controlling species leaving the fuel cell

Greek Letters

α_1 = width of the air channel (Figure 1), ft.

α_2 = thickness of the electrode-electrolyte assembly (Figure 1), ft.

α_3 = width of the fuel channel (Figure 1), ft.

β_0 = intercept of linearized polarization curve in Equation (5), volts

β_1 = slope of linearized polarization curve in Equation (5), ohm-sq. ft.

θ = dimensionless time = $\dfrac{k\tau}{\rho_{pa}c_{pa}a_3^2}$

ρ = density of gas, lb./cu. ft.

ρ_{pa} = density of pack, lb./cu. ft.

σ = electrical conductivity of the electrolyte-electrode assembly, mho/ft.

τ = time, hr.

ψ = rate of heat generation per unit volume of pack, B.t.u./hr.-cu. ft.

ψ_o = intercept of the rate of heat dissipation per unit total volume vs. temperature curve defined by Equation (23), B.t.u./hr.-cu. ft.

ψ_1 = slope of the rate of heat dissipation per unit total volume vs. temperature curve, defined by Equation (23), B.t.u./hr.-cu. ft.-°F.

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Scale-Up for Viscoelastic Fluids

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A suggestion is made as to how one might proceed to scale up a process involving an arbitrary viscoelastic fluid. The discussion is based upon Noll's theory of simple fluids. Turbulent flow in an infinitely long pipe is used as an illustration.

Many commercial processes involve viscoelastic* fluids from polymers and polymer solutions to food products. The behavior of these fluids is generally so complex that an accurate theoretical description of the flow in a given geometry may be out of the question. When the design engineer is faced with such a situation, one answer might be a model study, the results of which are correlated by dimensional analysis.

The most useful correlation of experimental data would be one valid for all fluids. This would permit the prediction of the results for one fluid in a given geometry on the basis of previous experiments with perhaps another fluid in a geometrically similar situation. Upon reflection, one realizes that to be able to do this it would be necessary to have a specific form of constitutive equation (an equation by means of which stress can be predicted from a knowledge of deformation) that would be valid for all fluids. Though many specific constitutive equations have been proposed, as yet none have been shown to be valid for more than a few fluids over a limited range of stress.

The next most useful scheme would allow one to predict the behavior of the fluid in the full-scale apparatus on the basis of data for the same fluid in a geometrically similar model. In what follows such an approach is out-

* Viscoelastic is used in the sense that the materials obey neither of the classical linear relations: Newton's law of viscosity and Hooke's law of elasticity.

lined. In order that attention can be focused on the treatment of viscoelasticity, the isothermal flow of a single homogeneous fluid is discussed. No free surfaces (fluid-fluid phase interfaces) are allowed. The fluid is specified to be incompressible, since viscoelasticity is most commonly observed in liquids. It is assumed that the behavior of the fluid may be described by Noll's theory of simple fluids (*1* to *6*).

THE NOLL SIMPLE FLUID

In some arbitrary, fixed, rectangular, Cartesian coordinate system $x_i (i = 1, 2, 3)$, the motion that took place in the material at all prior times $t + s$ $(-\infty < s \leq 0)$ may be described by specifying as a function of time, $t + s$, the coordinates X_i (in perhaps some other rectangular, Cartesian coordinate system) of the material that we find around the position $\mathbf{x}$ at time t.

$$X_i = X_i (\mathbf{x}, t + s) \tag{1}$$

We term $\partial X_i/\partial x_j$ the relative deformation gradient and

$$c_{ij}(t + s) = \sum_{m=1}^{3} \sum_{n=1}^{3} \delta_{mn} \frac{\partial X_m}{\partial x_i} \frac{\partial X_n}{\partial x_j} \tag{2}$$

the relative right Cauchy-Green strain tensor.

All constitutive equations for stress that have found any success at all have satisfied what Noll (*1*) has termed the principle of determinism; that is, the stress at the position $\mathbf{x}$ at time t is determined by the history of the motion of the material that is within an arbitrarily small neighborhood of $\mathbf{x}$ at time t. A *simple* material is defined by Noll to be one for which the stress t_{ij} at $\mathbf{x}$ and at time t is determined by the history of the relative deformation gradient, $\partial X_i/\partial x_j$, for the material that is in an arbitrarily small neighborhood of $\mathbf{x}$ at t. He defines an incompressible *simple fluid* as one for which the stress t_{ij} at $\mathbf{x}$ and at time t is specified within an indeterminant pressure p by the history of the relative right Cauchy-Green strain tensor for the material that is within an arbitrarily small neighborhood of $\mathbf{x}$ at t.

$$t_{ij} + p\delta_{ij} = \frac{\mu}{\tau} \overset{\circ}{H}{}^{*}_{\sigma=-\infty} {}_{ij} \quad [c_{km}(t + \sigma\tau)] \tag{3}$$

Here we follow Truesdell's discussion of the dimensional invariance of the definition of a simple material (*7*). The quantity $\overset{\circ}{H}{}^{*}_{\sigma=-\infty} {}_{ij}$ is a dimensionally invariant tensor-valued functional, that is, an operator that maps tensor-valued functions into tensors; the constants μ and τ, defined by Truesdell (*7*), are discussed below.

By using the definition of an incompressible simple fluid, Coleman and Noll (*2* to *6*) (see also *8*) have been able to show that there are at least two classes of flows that are dynamically possible for every simple fluid. The first class, sometimes referred to as the *viscometric flows*, includes several geometries that are interesting to the experimentalist, such as flow through a tube. The behavior of a simple fluid in any one of these geometries is determined by three material functions. One of these expresses the relation between shear stress and shear rate; the other two give differences between the normal stress components as a function of shear rate. Equation (3) as it is written applies only to an incompressible simple fluid that exhibits a linear viscosity μ, termed the *natural viscosity*, at sufficiently small shear rates and that shows some normal stress effects (*9*). The *natural time* τ is a positive constant that reflects the magnitude of the normal stress effect in the limit of small shear rates.

BASIS FOR A CORRELATION

There are two ways in which one may apply dimensional analysis to achieve a correlation of experimental data, one of which is the Pi theorem (*10*). A possible disadvantage in the use of this theorem is that one must make an assumption as to which variables are to be considered.

Another approach (*11*, p. 107) is to observe that the model experiment is designed such that its boundary value problem is precisely the same as that for the full-scale process when all variables are put in a dimensionally invariant form. This means that any functional relationship between dimensionless parameters that one can deduce from the dimensionally invariant equations and boundary conditions to be satisfied in the model experiment, and that one can evaluate by using data from the model experiment must be valid for the full-scale process as well. This is the attack taken here.

Putting the Cauchy stress equation of motion

$$\rho[\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla \mathbf{v})] = (\nabla \cdot \mathbf{t}) + \rho \mathbf{f} \tag{4}$$

in a dimensionally invariant form has been the only obstacle to correlating the flow data for an isothermal incompressible homogeneous viscoelastic fluid. But this difficulty is removed as a result of Truesdell's (*7*) discussion of the dimensionally invariant form of Noll's theory of simple fluids. Let us assume that the external body force vector per unit mass, $\mathbf{f}$, is derivable from a potential (this is justified in most cases when the body force is gravity).

$$\mathbf{f} = -\nabla \phi \tag{5}$$

After substituting Equations (3) and (5) into Equation (4), and after casting all variables in a dimensionally invariant form, we have

$$N_1 \partial \mathbf{v}^*/\partial t^* + (\mathbf{v}^* \cdot \nabla^* \mathbf{v}^*) = -\nabla^* P^* + N_2 \left(\nabla^* \cdot \overset{\circ}{\mathbf{H}}{}^{*}_{\sigma=-\infty} [\mathbf{c}\,(t + \tau\sigma)] \right) \tag{6}$$

where

$$\mathbf{v}^* = \mathbf{v}/V, \quad t^* = t/\tau, \quad P^* = (p + \rho\phi)/\rho V^2 \tag{7}$$

$$N_1 = R/(\tau V), \quad N_2 = \mu/(\tau \rho V^2) \tag{8}$$

The dimensionally invariant functional $\overset{\circ}{H}{}^{*}_{\sigma=-\infty}$ appearing in Equation (6), in general, is characteristic of a specific fluid. For this reason when one uses this approach, it is necessary to use the same fluid in the model experiments as that which one desires to use in the full-scale process.

An illustrative example may serve to clarify these ideas.

EXAMPLE: TURBULENT FLOW IN AN INFINITELY LONG TUBE

In addition to Equation (6), the flow must satisfy the equation of continuity, which in a dimensionally invariant form becomes

$$(\nabla^* \cdot \mathbf{v}^*) = 0 \tag{9}$$

Since this is an illustrative example, the boundary conditions will be described in detail. As an initial condition we assume the flow to be laminar. The laminar velocity distribution is known (*2*, *3*); in a dimensionally invariant form it becomes

$$\text{For } t \leq 0: \quad v_r^* = v_\theta^* = 0,$$

$$v_z^* = \frac{N_1}{N_3} \int_{N_3 r^*}^{N_3} \tau^{*-1}(\eta)\, d\eta \tag{10}$$

where

$$N_3 = a\tau R/(2\mu) \tag{11}$$

The function $\tau^{*-1}(N_3\tau^*)$ is the dimensionally invariant inverse of one of the three material functions that describe the behavior of a Noll simple fluid in a viscometric flow (2 to 5). Since the characteristic velocity is the average axial component of velocity, the groups N_1 and N_3 are related by

$$1 = \frac{N_1}{(N_3)^3} \int_0^{N_3} \eta^2 \tau^{*-1}(\eta)\, d\eta \tag{12}$$

For an initial period of time $0 < t \leq t_1$, a portion of the solid wall vibrates with a small amplitude b and a frequency ω. In detail, this means

$$\text{for } 0 < t^* \leq t_1^*, \quad \theta_1 < \theta < \theta_2, \quad z_1^* < z^* < z_2^*,$$

$$r^* = 1 + \frac{b}{R} \sin N_4\, t^*:$$

$$v_r^* = N_1\, N_4 \frac{b}{R} \cos N_4\, t^*, \quad v_\theta^* = v_z^* = 0 \tag{13}$$

$$\text{for } 0 < t^* \leq t_1^*, \quad 0 \leq \theta < \theta_1 \quad \text{or}$$

$$\theta_2 < \theta \leq 2\pi, \quad z_1^* < z^* < z_2^*, \quad r^* = 1: \quad \mathbf{v}^* = 0 \tag{14}$$

$$\text{for } 0 < t^* \leq t_1^*, \quad z^* < z_1^* \quad \text{or} \quad z^* > z_2^*,$$

$$r^* = 1: \quad \mathbf{v}^* = 0 \tag{15}$$

where

$$N_4 = \omega\tau \tag{16}$$

For all remaining time, the walls of the tube are fixed

$$\text{for } t^* > t_1^*, \quad r^* = 1: \quad \mathbf{v}^* = 0 \tag{17}$$

A further restriction on the flow is that the volume rate of flow through the tube be a constant with respect to time.

for all t^*:

$$1 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 v_z^*\, r^*\, dr^*\, d\theta \tag{18}$$

Equations (6) and (9) through (18) suggest that

$$\mathbf{v}^* = \mathbf{v}^*\left(N_1, N_2, N_4, \theta_1, \theta_2, z_1^*, z_2^*, \frac{b}{R}, t_1^*, r^*, \theta, z^*, t^*\right) \tag{19}$$

But for $t^* >> t_1^*$, we suspect that the influence of the initial disturbance must be negligible. For that matter, we suspect that the influence of any very small disturbance must be negligible once the flow has become turbulent. Therefore we may write

$$\text{for } t^* >> t_1^*: \quad \mathbf{v}^* = \mathbf{v}^*(N_1, N_2, r^*, \theta, z^*, t^*) \tag{20}$$

For design purposes, we should like to mention the friction factor (*11*, p. 181)

$$f = \frac{1}{\pi RL\, \rho V^2} \int_0^L \int_0^{2\pi} -t_{rz}\big|_{r=R} R\, d\theta\, dz \tag{21}$$

This becomes in terms of dimensionless variables

$$f = \frac{N_2}{\pi\, L/R} \int_0^{L/R} \int_0^{2\pi} -t^*_{rz}\big|_{r^*=1}\, d\theta\, dz^* \tag{22}$$

It is suggestive that a wall shear stress averaged over θ is independent of distance along a section of tube far removed from either end, so that Equation (22) reduces to

$$f = \frac{N_2}{\pi} \int_0^{2\pi} -t^*_{rz}\big|_{r^*=1}\, d\theta \tag{23}$$

Integrating the definition of velocity

$$v^i = \frac{dX^i}{d(t+s)} \tag{24}$$

we have

$$X^{*i} - x^{*i} = \frac{1}{N_1} \int_{t^*}^{t^*+\sigma} v^{*i}\, d(t^* + \sigma) \tag{25}$$

Equations (2), (3), (20), (23), and (25) indicate that for the usual experimental conditions under which one would study turbulent flow through a tube in the absence of end effects

$$f = f(N_1, N_2) \tag{26}$$

Since only a functional relationship for a single fluid is being considered we need not take into account μ and τ†

$$f = \bar{f}\left(\frac{R}{V}, V^2\right) \tag{27}$$

For a particular Newtonian fluid we know that the family of curves suggested by Equation (27) may be reduced to a single curve describing the friction factor as a function of flow rate for all tube diameters; we plot f vs. $RV\left(=\frac{RV^2}{V}\right)$. Shaver and Merrill (*12*) and Dodge and Metzner (*13*) have shown that many fluids may be reasonably well described by the power model; for such a fluid Equation (27) collapses to a single curve for f as a function of $R^n\, V^{2-n}$, where n is the exponent in the power model.

Unfortunately, not all materials are so well behaved. Savins (*14*) discusses fluids that are reasonably well described by the power model, at least to the extent of correlating the friction factor in apparently laminar flow. Yet these fluids exhibit a *diameter effect* in turbulent flow; that is, the friction factor cannot be represented by a single combination of R/V and V^2. This is entirely unexpected from the viewpoint of the power model. Yet this is not very surprising, since there are obvious reasons why the power model does not tell the entire story (*15*, p. 245). In view of Equation (27) deduced from the Noll theory, such a diameter effect is entirely reasonable.

The problem of finding specific models for fluid behavior that allow the correlation of tube flow data for classes of materials is not an easy one. The discussions of Savins (*14*) and Meter (*16*) indicate that the class of fluids has not been established for which the correlations of Shaver and Merrill (*12*) and Dodge and Metzner (*13*) are applicable. Meter's correlation (*16*), which assumes that the behavior of the fluid may be represented by a four-constant model for stress as a function of the rate-of-deformation (*17*), is based upon data for seven aqueous solutions of Natrosol hydroxyethylcellulose; it does not simplify to the limiting case of extremely dilute solutions. Thomas's correlations (*18*, *19*) are based upon the Bingham Plastic model and appear to be limited to certain suspensions. It may be that until a more generally applicable correlation is available, the simplest approach to the scale-up of turbulent pipe flow data for a particular viscoelastic fluid may be the correlation suggested in Equation (27).

NOTATION

- a = driving force per unit area for flow through a tube
- b = amplitude of vibration in Equation (13)
- c_{ij} = the relative right Cauchy-Green strain tensor, Equation (2)

† A special case of Equation (27) is proposed intuitively by Bowen (*20*) [see his Equations (39) and (40)].

f = friction factor, defined by Equation (21)
$\mathbf{f}$ = external body force vector per unit mass
L = length of section of pipe considered in Section 4
N_1, N_2, N_3, N_4 = dimensionless groups defined by Equations (8), (11), and (16), respectively
p = pressure
P^* = defined by Equation (7)
r = radial cylindrical coordinate
R = characteristic length. Radius of tube in Section 4
t = time
$\mathbf{t}$ = stress tensor
t^*_{rz} = dimensionless physical component of stress tensor in cylindrical coordinates
$\mathbf{v}$ = velocity vector
V = characteristic scalar having the dimensions of velocity. The average axial component of velocity in Section 4
$x_i(\mathbf{x})$ = coordinates in an arbitrary, fixed, rectangular, Cartesian frame of reference
$X_i(\mathbf{X})$ = coordinates (in perhaps some other rectangular, Cartesian frame) of the material which we find around the position $\mathbf{x}$ at time t
z = axial cylindrical coordinate

Greek Letters

δ_{ij} = Kronecker delta
θ = cylindrical coordinate
μ = natural viscosity for a Noll simple fluid (7)
ρ = density
τ = natural time for a Noll simple fluid (7)
$\tau^{*-1}(N_3\tau^*)$ = dimensionally invariant (7) inverse of one of the three material functions that describe the behavior of a Noll simple fluid in a viscometric flow (2 to 5)
ϕ = potential in terms of which the body force vector per unit mass is defined, Equation (5)
ω = frequency of vibration in Equation (13)

Superscript

* = a dimensionally invariant quantity. All lengths, velocities, times, and stresses are made dimensionless with respect to R, V, τ, and μ/τ, respectively
∇ = del operator

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Mass Transfer at High Mass Fluxes:

Part I. Evaporation at the Stagnation Point of a Cylinder

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An experimental method for measuring mass transfer rates at high mass fluxes is developed for the case of evaporation at the stagnation point of a cylinder. Evaporation of carbon tetrachloride at the stagnation point was measured and determined to be consistent with theoretical predictions. It was observed that water exhibits a significant interfacial resistance to evaporation at high transfer rates. Evaporation coefficients based on the discrepancy between experimental determinations and theoretical expectations were found to be a strong function of temperature and to be consistent with the majority of reported measurements made at lower temperatures. The experimental method should be useful for application to other configurations and flow conditions for studying mass transfer at high mass fluxes.

Considerable experimental and theoretical efforts have been directed to the study of mass transfer phenomena. Unfortunately, few experimental studies investigating the transport process at high concentration levels and at high transport rates have been reported. Cairns and Roper (5) studied simultaneous heat and mass transfer in a wetted wall column using the air-water system at conditions where high concentration levels were achieved over much of the apparatus. Westkaemper and White (21) investigated the effect of concentration level on the evaporation of carbon tetrachloride from a free liquid surface into air in turbulent flow. More recently, Schulman and Delaney (19) measured the evaporation of carbon tetrachloride into air in a packed column as a function of concentration

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